$$\frac{K'(T_1 - T_m)\exp(-h^2k/k')}{k'^{1/2}\,\mathrm{erf}\,(h\sqrt{k/k'})} - \frac{KT_m \exp(-h^2)}{k^{1/2}\,\mathrm{erfc}\,h} = \pi^{1/2}k^{1/2}L\rho h \quad (10)$$

Here K is the conductivity, k the diffusivity, L the latent heat of fusion, primed quantities refer to the thermal constants of the molten solid, and the constant h is the root of the transcendental equation (10). If we introduce dimensionless coordinates $\xi = cx/4h^2k$ and $\tau = c^2t/4h^2k$, then the boundary curve becomes $\xi = \tau^{1/2}$, and the characteristic that separates regions R_1 and R_2 becomes $\xi = \tau + {}^1/_4$. The steps outlined in Eqs. (6) to (8) may now be carried out. The result is as follows:

$$\sigma_x = -{}^1/_2\rho\kappa T_m(\mathrm{erfc}\,h)^{-1}[\exp h^2(\tau - \xi)\cdot \mathrm{erfc}\,h(\xi\tau^{-1/2} - 2\tau^{1/2}) + \exp h^2(\tau + \xi).\,\mathrm{erfc}\,h(\xi\tau^{-1/2} + 2\tau^{1/2})] \quad \text{for } 0 < \tau < {}^1/_4,\ \xi > \tau^{1/2} \text{ and for } \tau > {}^1/_4,\ \xi > \tau + {}^1/_4 \quad (11)$$

$$\sigma_x = -{}^1/_2\rho\kappa T_m(\mathrm{erfc}\,h)^{-1}\{\exp h^2(\tau - \xi)\cdot[\mathrm{erfc}\,h(2\tau^{1/2} - \xi\tau^{-1/2}) - \mathrm{erfc}\,h\sqrt{1 - 4(\xi - \tau)}] + \exp h^2(\tau - \xi + 1 + \sqrt{1 - 4(\xi - \tau)})\cdot \mathrm{erfc}\,h(2 + \sqrt{1 - 4(\xi - \tau)}) - \exp h^2(\tau - \xi)\,\mathrm{erfc}\,h(\xi\tau^{-1/2} + 2\tau^{1/2})\} \quad \text{for } \tau > {}^1/_4,\ \tau^{1/2} < \xi < \tau + {}^1/_4 \quad (12)$$

The jump in the value of stress which propagates with velocity c is given by

$$\Delta\sigma_x = -{}^1/_2\rho\kappa T_m(\mathrm{erfc}\,h)^{-1}[\exp(-h^2/4) + \exp(3h^2/4)\cdot \mathrm{erfc}\,(2h)] \quad (13)$$

We note that this discontinuity in the value of stress increases with increasing h. The same effect is present in the expressions for σ_x as given in Eqs. (11) and (12). For a given material the coefficient $\rho\kappa T_m$ that appears in Eqs. (11) to (13) is a constant, whereas from Eq. (10) h increases as T_1 increases. This clearly demonstrates that under these circumstances the effect of inertia becomes important for all time, and its neglect is not justified. It may easily be verified that the quasi-static solution of the problem is given by $e = \kappa c^{-2}T$, $\sigma_x \equiv 0$, and $\sigma_y = \sigma_z = -\rho\kappa T$.

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Interface Stability in a Nonuniform Acceleration Field

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July 11, 1962

THE STABILITY of two fluids separated by an initially plane interface and subject to a constant, normal acceleration field was considered by Taylor.[1] Corrections were introduced later to Taylor's analysis to account for the viscous and cohesive forces.[2]

The interface stability of two fluids in motion parallel to the common interface in a normal constant gravitational field was first studied by Helmholtz.[3, 4]

In some cases of practical importance the fluids are exposed to the action of nonuniform body-force distributions. For example, if a binary fluid with a distinguishable separation surface is in a vortex-type motion, the radial variation of the azimuthal velocity in the potential-flow region causes a spatially nonuniform centripetal acceleration field on both sides of the interface. The inclusion of viscosity effects would increase this nonuniformity.

The question is: how does the acceleration-field distribution affect the stability of the separation surface of two fluids?

The aim of this note is to show that the interface stability does not depend upon the spatial distribution of the acceleration field; the Helmholtz and Taylor stability criteria are directly applicable if the direction and the magnitude of the acceleration at the interface are known.

For brevity, only the invariance of the Taylor stability criterion will be shown here. Similar considerations can be applied also to Helmholtz's analysis.

Generalization of Taylor's Analysis

For the notation and the model considered here the reader is referred to Refs. 1 and 2.

The linearized equations describing the disturbed flow field are

$$\left.\begin{aligned} \partial u/\partial x + \partial v/\partial y &= 0 \\ \partial u/\partial t + (1/\rho)(\partial p/\partial x) &= 0 \\ \frac{\partial v}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial y} + g(y) &= 0 \end{aligned}\right\} \quad (1)$$

and the interface is given by $y = \eta(x, t)$. Here, u and v are velocity components generated by the initially small disturbances; $g(y)$ is an arbitrary function of the coordinate perpendicular to the interface and it describes the body-force distribution throughout the fluid masses.

The above equations can be satisfied by

$$\left.\begin{aligned} u = \partial\phi/\partial x \quad v &= \partial\phi/\partial y \\ p = p_0 - \rho\int_0^y g(y)dy &- \rho\frac{\partial\phi}{\partial t} \end{aligned}\right\} \quad (2)$$

where p_0 is the mean pressure at the interface. Let us assume now

$$\left.\begin{aligned} \phi_1 &= A_1\exp[-ky + \sigma t + ik'x] \\ \phi_2 &= A_2\exp[ky + \sigma t + ik'x] \end{aligned}\right\} \quad (3)$$

where the subscripts 1, 2, refer to the upper and lower fluids, respectively.

The condition of pressure continuity at the disturbed interface yields the following expression:

$$\int_0^{y=\eta}[\rho_2g_2(y) - \rho_1g_1(y)]dy = \sigma[\rho_1A_1e^{-k\eta} - \rho_2A_2e^{k\eta}]\exp[\sigma t + ik'x] \quad (4)$$

If the initial disturbance amplitude is small compared with the wavelength, the time derivative of Eq. (4) can be written as

$$[\rho_2g_2(\eta) - \rho_1g_1(\eta)](\partial\eta/\partial t) = \sigma^2(A_1\rho_1 - A_2\rho_2)\exp[\sigma t + ik'x] \quad (5)$$

But

$$g_2(\eta) = g_1(\eta) = g_0 \quad (6)$$

where g_0 is the magnitude of the body forces at the interface.

The condition of continuous velocity at the interface for small initial-amplitude values yields

$$A_2 = -A_1 = -A \quad (7)$$

If one applies now the kinematic surface condition[5]

$$(\partial\eta/\partial t) + u(\partial\eta/\partial x) = v \quad (8)$$

and neglects the higher-order convective term,

$$\frac{\partial\eta}{\partial t} \simeq \frac{\partial\phi}{\partial y} = -kA\exp(\sigma t + ik'x) \quad (9)$$

The combination of Eqs. (5), (6), (7), and (9) yields an expression for the damping factor σ:

$$\sigma^2 = -kg_0[(\rho_2 - \rho_1)/(\rho_2 + \rho_1)] \quad (10)$$

The result is identical with that obtained for a constant acceleration field g_0.[1]

Thus, the stability of the interface depends upon the body forces

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acting at the interface, but it is not affected by the g-field distributions throughout the fluid masses.

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Librational Dynamic-Response Limits of Gravity-Gradient Satellites

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September 12, 1962

IT IS KNOWN that the gradient of gravitational-attraction forces generates a restoring couple on a satellite slightly displaced in angular orientation from its equilibrium attitude.[1] The positive restoring moment thus obtained provides a stabilizing influence in purely passive configurations, of interest in such applications as communications reflectors. The weakness of the gravity gradient, however, is equivalent to the use of a very soft spring when displacements are small. This explains excessively long periods of natural oscillation and consequent serious problems of attitude control. Detailed analysis reveals a slight additional stabilizing couple due to inertia and permits appraisal of response characteristics for a wide class of satellite configurations.

An elementary vector identity suggests a compact representation of gravity torque by means of a vector potential

$$\mathbf{W} = -G\int(\mathbf{r}/R)dm$$

where G is the usual product of earth mass times universal gravitational constant, $\mathbf{r}$ is position vector measured from center of moments (which we take as satellite mass center) to a point of the satellite, R is the distance from earth center to the same point, and integration is carried out over all mass points of the satellite. Gravitational torque moment is then given by the vector curl operation as

$$\mathbf{M}_0 = \nabla \times \mathbf{W} \tag{1}$$

where nonzero terms originate only in the term R, which may be written as

$$R^2 = (R_0 + z)^2 + x^2 + y^2$$

R_0 being the distance of satellite mass center from earth center, z is measured outward along the same radial, y is directed oppositely to orbital velocity, and x is such as to form a right-handed Cartesian system. Since R_0 is much greater than x, y, or z, the moment $\mathbf{M}_0$ is given by

$$\mathbf{M}_0 = (3G/R_0^3)[\mathbf{i}\int yzdm - \mathbf{j}\int xzdm] \tag{2}$$

$\mathbf{i}$, $\mathbf{j}$ denoting unit vectors in x, y directions (the same result easily obtains by direct integration of elementary force moments without use of potential $\mathbf{W}$). When principal inertia axes are displaced through infinitesimal angles α, β, γ about x, y, z axes and corresponding inertia moments are denoted by A, B, C, respectively, Eq. (2) is linearly approximated as

$$\mathbf{M}_0 = -(3G/R_0^3)\{\alpha(B - C)\mathbf{i} + \beta(A - C)\mathbf{j}\} \tag{3}$$

With $B > C$, $A > C$, both components have negative signs indicating restoring torques and static stability. These inequalities are appropriate for elongated figures in the radial direction, for which maximum gravity-gradient forces are realized. It is evident that 90° rotations interchange pairs of A, B, C values, and consequent sign reversals indicate instabilities. Expressions (2) or (3) determine the rate of change of total satellite moments of momentum; this has the form

$$\ddot{\alpha}A\mathbf{i}_1 + [\ddot{\beta}B - \dot{\gamma}\Omega B + (\dot{\gamma} + \beta\Omega)\Omega(A - C)]\mathbf{i}_2 + [\ddot{\gamma}C + \dot{\beta}\Omega C - (\dot{\beta} - \gamma\Omega)\Omega(A - B)]\mathbf{i}_3 \tag{4}$$

when referred to principal-axis coordinates. Time derivatives are indicated by dots and orbital angular velocity Ω is introduced, given by

$$\boldsymbol{\Omega} = \Omega\mathbf{i}_j \quad \Omega^2 = G/R_0^3$$

The separate components of Eqs. (4) reveal dynamic coupling ($\dot{\beta}, \dot{\gamma}$ terms) with orbital motion and centrifugal couples, stabilizing when $A > C$, $A > B$. Within present small angle approximations $\mathbf{i} = \mathbf{i}_1$ and $\mathbf{j} = \mathbf{i}_2$, so that (3) and (4) together determine orientation in space by means of three second-order scalar equations for α, β and γ. Thees equations exhibit the coupling of β and γ motions in all cases except when the condition

$$A - B - C = 0 \tag{5}$$

is satisfied, which is of much less physical interest than the case of the prolate axially symmetric figure for which

$$A = B > C \tag{6}$$

The α motion is in any case uncoupled from the other two modes and is characterized by a natural frequency given by

$$\omega_\alpha = \sqrt{3}\Omega[(B - C)/A]^{1/2} \tag{7}$$

corresponding to less than two cycles per orbit. The maximum value occurs in the limit $C \rightarrow 0$, i.e., for configurations everywhere close to the z-axis (for which signal reflection cross-section is also most reduced). From the defining integrals of the moments of inertia it is also seen that the frequency of α motion (oscillations confined to orbital plane) is reduced for the asymmetric case which does not conform to the equality in (6).

The coupled equations for β and γ motion are

$$\begin{aligned} B\ddot{\beta} + \Omega(A - B - C)\dot{\gamma} + 4\Omega^2(A - C)\beta &= 0 \\ C\ddot{\gamma} - \Omega(A - B - C)\dot{\beta} + \Omega^2(A - B)\gamma &= 0, \end{aligned} \tag{8}$$

which bear a close resemblance to those for Foucault's pendulum.[2] This is natural, of course, since our configuration is also a "pendulum," supported by orbital centrifugal force. As in Foucault's case, the two motions are 90° out of phase with each other. Differences appear when the final terms are compared: two distinct frequencies are present in our case if $A \neq B$, and this again signifies reduction in values of natural frequencies. In no case is the system frequency greater than twice the orbital value.

It is therefore seen, even when the centrifugal restoring couple is included, that gravity-gradient passive satellite stability is at best marginal. Large-amplitude displacements are thus invited which lead directly to definite instability. This happens most easily, moreover, for those elongation configurations which give the best response when in their orientation of stable equilibrium. The practical requirement for added system complexity is thus shown.

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